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Correlated Fractal Percolation and the Palis Conjecture

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Abstract Let F_1 and F_2 be independent copies of one-dimensional correlated fractal percolation, with almost sure Hausdorff dimensions $\dim_{\mathrm{H}}(F_1)$ and $\dim_{\mathrm{H}}(F_2)$. Consider the following question: does $\dim_{\mathrm{H}}(F_1) + \dim_{\mathrm{H}}(F_2) > 1$ imply that their algebraic difference $F_1 - F_2$ will contain an interval? The well known Palis conjecture states that 'generically' this should be true. Recent work by Kuijvenhoven and the first author (Dekking and Kuijvenhoven in J. Eur. Math. Soc., to appear) on random Cantor sets cannot answer this question as their condition on the joint survival distributions of the generating process is not satisfied by correlated fractal percolation. We develop a new condition which permits us to solve the problem, and we prove that the condition of Dekking and Kuijvenhoven (J. Eur. Math. Soc., to appear) implies our condition. Independently of this we give a solution to the critical case, yielding that a strong version of the Palis conjecture holds for fractal percolation and correlated fractal percolation: the algebraic difference contains an interval almost surely if and only if the sum of the Hausdorff dimensions of the random Cantor sets exceeds one.

Keywords Palis conjecture · Algebraic difference · Cantor sets · Correlated fractal percolation · Branching processes · Criticality

1 Introduction

In this paper we consider a natural class (called correlated fractal percolation) of random Cantor sets with dependence, as opposed to the independent case, which is know as fractal percolation or Mandelbrot percolation. Two and three dimensional versions of both types of sets have occurred before in the literature, especially as a modeling tool, see e.g., [6], where the dependent case is called the 'homogeneous algorithm', and the independent case the 'heterogeneous algorithm' (see Fig. 1 left, respectively right for an illustration of these

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Fig. 1 *Left*: Two-dimensional 7 out of 9 correlated fractal percolation with $\mu(\emptyset) = 0$. *Middle*: Two-dimensional 8 out of 9 correlated fractal percolation with $\mu(\emptyset) = \frac{1}{8}$. *Right*: Ordinary two-dimensional fractal percolation with $p = 7/9$

two processes by two realizations). In [3] they are called 'constrained curdling', respectively 'canonical curdling'. All this work has its roots in the seminal paper [4].

Our main goal is to answer the question whether or not an interval occurs in the algebraic difference of two independent random Cantor sets from the correlated fractal percolation class. A complete answer is given in Theorem 3 in Sect. 5.

We also call correlated fractal percolation m out of M percolation (cf. Sect. 2.2), where m is an integer with $1 \le m \le M$. It will appear that the transition from no interval to interval lies at values of $m \approx \sqrt{M}$. The combinatorial Lemma 6 lies at the basis for a solution of all cases, except the case $m = \sqrt{M+1}$, which is a tough nut to crack (Lemma 7).

The key idea to obtain these results is that we introduce a new condition on the survival distributions, which improves on the condition given in [2]. As a bonus, this gives a more general and more simple proof of the basic theorem (Theorem 2). It is more simple since we do not need the combinatorial 'color lemma' of [1] and [2], nor the irreducibility condition of [2].

2 Differences of Random Cantor Sets

Here we will introduce M-adic random Cantor sets and their differences, and the main result (Theorem 1) from [2] regarding the Palis conjecture, with a rough sketch of the proof. Finally we describe higher order Cantor sets which are particularly useful to obtain a more complete characterization from Theorem 1.

2.1 M-adic Random Cantor Sets

An M-adic random Cantor set F is constructed using the following mechanism: take the unit interval and divide it into M subintervals of equal length. Each of those subintervals corresponds to a letter in the alphabet $\mathbb{A} = \{0, \dots, M-1\}$. It will be convenient to consider $\mathbb{A}$ as an Abelian group with addition. So for instance if $M = 6$ we have $5+3 = 2$. Now define a *joint survival measure* μ on $2^{2^{\mathbb{A}}}$. It is determined by its values $(\mu(A))$ on the singletons $A \subset \mathbb{A}$. According to this distribution we choose which subintervals are kept and which are discarded. Then in each next construction step, each of the surviving subintervals is again divided in M subintervals of equal length, of which a subset survives according to the distribution μ.

More formally, we consider the space of $\{0, 1\}$-labeled M-adic trees $\{0, 1\}^{\mathcal{T}}$, where we label each node $i_1 \dots i_n \in \mathcal{T}$ with $X_{i_1 \dots i_n} \in \{0, 1\}$.

The probability measure $\mathbb{P}_\mu$ on this space is defined by requiring that $\mathbb{P}_\mu(X_\emptyset = 1) = 1$ (where $\emptyset$ is the root of $\mathcal{T}$), and that for all $i_1 \dots i_n \in \mathcal{T}$ the random sets

$$\{i_{n+1} \in \mathbb{A} : X_{i_1 \dots i_n i_{n+1}} = 1\}$$

are independent and identically distributed according to μ. We let $\mathcal{T}_n$ denote the set of nodes at level n, and for any $\underline{i}_n = i_1 \dots i_n$ from $\mathcal{T}_n$ we define the associated M-adic interval by

$$I_{i_1 \dots i_n} := \left[\frac{i_1}{M} + \cdots + \frac{i_{n-1}}{M^{n-1}} + \frac{i_n}{M^n}, \frac{i_1}{M} + \cdots + \frac{i_{n-1}}{M^{n-1}} + \frac{i_n + 1}{M^n} \right].$$

The n-th level approximation F^n of the random Cantor set is a union of such n-th level M-adic intervals selected by the sets S_n defined by

$$S_n = \{i_1 \dots i_n : X_{i_1} = X_{i_1 i_2} = \cdots = X_{i_1 \dots i_n} = 1\}.$$

The random Cantor set F is

$$F = \bigcap_{n=1}^{\infty} F^n = \bigcap_{n=1}^{\infty} \bigcup_{i_1 \dots i_n \in S_n} I_{i_1 \dots i_n}.$$

The *marginal probabilities* p_i of μ are defined for $i \in \mathbb{A}$ by

$$p_i := \sum_{X \subseteq \mathbb{A} : i \in X} \mu(X). \tag{1}$$

We start with the definition of the class of random Cantor sets which we will take into consideration.

2.2 Correlated Fractal Percolation

From now on we will consider one-dimensional fractal percolation.

Definition 1 Suppose μ assigns the *same* positive probability to all subsets of $\mathbb{A}$ with m elements for some fixed integer $1 \le m \le M$, and that μ assigns probability zero to all other non-empty subsets of $\mathbb{A}$. If $p := (1 - \mu(\emptyset))\frac{m}{M}$ then we call this (m, M, p)-percolation.

We can compute the marginal probabilities of (m, M, p)-percolation as follows. Let X be a subset of $\mathbb{A}$, chosen according to the joint survival distribution μ. The probability that X is non-empty is $1 - \mu(\emptyset)$. Given that X is non-empty, the probability that a fixed $k \in \mathbb{A}$ belongs to X equals m/M. It follows that for $k \in \mathbb{A}$ the marginal probability p_k is given by

$$p_k = (1 - \mu(\emptyset))\frac{m}{M} = p,$$

which is exactly the reason why we defined (m, M, p)-percolation by requiring that $p = (1 - \mu(\emptyset))m/M$. Because $0 \le \mu(\emptyset) \le 1$, (m, M, p)-percolation is only defined for $0 \le p \le \frac{m}{M}$. From now on we will assume that $p > 0$ and $m > 0$, since giving the empty set probability one does not yield the most exciting situation.

2.3 Algebraic Differences of Sets

The *algebraic difference* $F_1 - F_2$ of the sets F_1 and F_2 is defined by

$$F_1 - F_2 = \{x - y : x \in F_1,\ y \in F_2\}.$$

The well known Palis conjecture [5] states that 'generically' $\dim_{\mathrm{H}}(F_1) + \dim_{\mathrm{H}}(F_2) > 1$ should imply that the algebraic difference $F_1 - F_2$ will contain an interval.

This question is considered in [1] and [2] for two M-adic random Cantor sets F_1 and F_2 with the same M but not necessarily the same joint survival distribution.

One can distinguish between joint survival distributions selecting intervals independently and joint survival distributions not having this property. In the independent case, the problem is somewhat less complicated, but still far from trivial. Intervals are selected and discarded independently if and only if the joint survival distribution satisfies for all $X \subseteq \mathbb{A}$ the equality

$$\mu(X) = \prod_{i \in X} p_i \prod_{i \notin X} (1 - p_i). \tag{2}$$

An important role in the answer to the main question is played by the *cyclic cross-correlation coefficients* (mostly simply called correlation coefficients)

$$\gamma_k := \sum_{i=0}^{M-1} q_i \, p_{i+k}, \quad \text{for } k \in \mathbb{A},$$

where (p_i) and (q_i) are the vectors of marginal probabilities of the joint survival distributions μ, respectively λ.

The result of [2] needs the following condition (which is satisfied in the independent case of (2)).

Condition 1 *A joint survival distribution* $(\mu(A))_{A \subseteq \mathbb{A}}$ *satisfies the* joint survival condition (JSC) *if it assigns positive probability to the marginal support* $\mathrm{Supp}_m(\mu)$ *of* μ, *which is defined by*

$$\mathrm{Supp}_m(\mu) := \bigcup \{X \subseteq \mathbb{A} : \mu(X) > 0\} = \{i \in \mathbb{A} : p_i > 0\}.$$

The following result of [2] generalizes the main theorem of [1].

Theorem 1 *Consider two independent random Cantor sets F_1 and F_2 whose joint survival distributions μ and λ both satisfy Condition 1, the* JSC.

(1) *If $\gamma_k > 1$ for all $k \in \mathbb{A}$, then $F_1 - F_2$ contains an interval a.s. on $\{F_1 - F_2 \neq \emptyset\}$.*
(2) *If $\gamma_k < 1$, $\gamma_{k+1} < 1$ for some $k \in \mathbb{A}$, then $F_1 - F_2$ contains no interval a.s.*

Obviously for (m, M, p)-percolation the JSC is not satisfied, unless we are in the case $m = M$, giving positive probability only to the full alphabet and the empty set (actually, this is ordinary fractal percolation, where intervals are discarded independently and the marginal probabilities p_k are all equal to p).

2.4 The Geometry of the Algebraic Difference

We will give in this subsection the tools and the notation introduced in [1] and [2].

Let $\phi : [0,1]^2 \to [-1,1]$ be given by $\phi(x,y) = y - x$, then $F_1 - F_2 = \phi(F_1 \times F_2)$. Thus $F_1 - F_2$ is defined on the product space of the probability spaces of F_1 and F_2. We will use $\mathbb{P} := \mathbb{P}_\mu \times \mathbb{P}_\lambda$ to denote the corresponding product measure and $\mathbb{E}$ to denote expectations with respect to this probability.

Let F_1 and F_2 be two independent M-adic random Cantor sets with joint survival distributions μ and λ, respectively. Denote by F_1^n and F_2^n their nth level approximations ($n \ge 0$) and define the following subsets of the unit square $[0,1]^2$:

$$\Lambda^n := F_1^n \times F_2^n, \quad n \ge 0, \qquad \Lambda := F_1 \times F_2 = \bigcap_{n=0}^{\infty} \Lambda^n.$$

Note that as $F_1^n \downarrow F_1$ and $F_2^n \downarrow F_2$, also $\Lambda^n \downarrow \Lambda$.

The Λ^n are unions of M-adic squares

$$Q_{i_1\ldots i_n, j_1\ldots j_n} := I_{i_1\ldots i_n} \times I_{j_1\ldots j_n},$$

with $i_1 \ldots i_n, j_1 \ldots j_n \in \mathcal{T}_n$ and $n \ge 0$.

Note that ϕ acts as a 45° projection on the x-axis. Similarly to [1] and [2] we scale and rotate the unit square over 45° counterclockwise, to rather see it as a 90° projection on $[-1,1]$. See Fig. 2 for a graphical representation of some of the squares Q and their ϕ-images. Here we denote the M-adic intervals $I_{i_1\ldots i_n}$ in $[0,1]$ by $I^R_{i_1\ldots i_n}$ (they are projections of squares in the right side of the tilted square), and define

$$I^L_{i_1\ldots i_n} = I^R_{i_1\ldots i_n} - 1,$$

for the M-adic intervals $I_{i_1\ldots i_n}$ in $[-1,0]$ (they come from the left side). The *columns* $C^U_{k_1\ldots k_n}$, where $U = L$ or $U = R$ are defined for each $k_1 \ldots k_n \in \mathcal{T}$ by

$$C^U_{k_1\ldots k_n} := \phi^{-1}\big(I^U_{k_1\ldots k_n}\big).$$

Note that any nth level M-adic square $Q_{i_1\ldots i_n, j_1\ldots j_n}$ is split into a 'left' and a 'right' triangle by the M-adic columns. These triangles are called L-triangles and R-triangles, and will be denoted by $L_{i_1\ldots i_n, j_1\ldots j_n}$ and $R_{i_1\ldots i_n, j_1\ldots j_n}$ respectively, for any $i_1 \ldots i_n, j_1 \ldots j_n \in \mathcal{T}$.

For all $U, V \in \{L, R\}$ and $\underline{k}_n \in \mathcal{T}$ we let

$$Z^{UV}(\underline{k}_n) := \#\big\{(\underline{i}_n, \underline{j}_n) : Q_{\underline{i}_n, \underline{j}_n} \subseteq \Lambda^n, V_{\underline{i}_n, \underline{j}_n} \subseteq C^U_{\underline{k}_n}\big\}$$

denote the number of level n V-triangles in $\Lambda^n \cap C^U_{\underline{k}_n}$. We also denote the total number of V-triangles in columns $C^L_{\underline{k}_n}$ and $C^R_{\underline{k}_n}$ together by

$$Z^V(\underline{k}_n) := Z^{LV}(\underline{k}_n) + Z^{RV}(\underline{k}_n),$$

for all $\underline{k}_n \in \mathcal{T}$. For example, in Fig. 2 we have $Z^R(01) = 1 + 2 = 3$.

An important observation is that an M-adic interval $I^U_{\underline{k}_n}$ is absent in $\phi(\Lambda^n)$ exactly when there are no triangles in the corresponding column $C^U_{\underline{k}_n}$ in Λ^n:

$$I^U_{\underline{k}_n} \not\subseteq \phi(\Lambda^n) \iff Z^{UL}(\underline{k}_n) = Z^{UR}(\underline{k}_n) = 0.$$

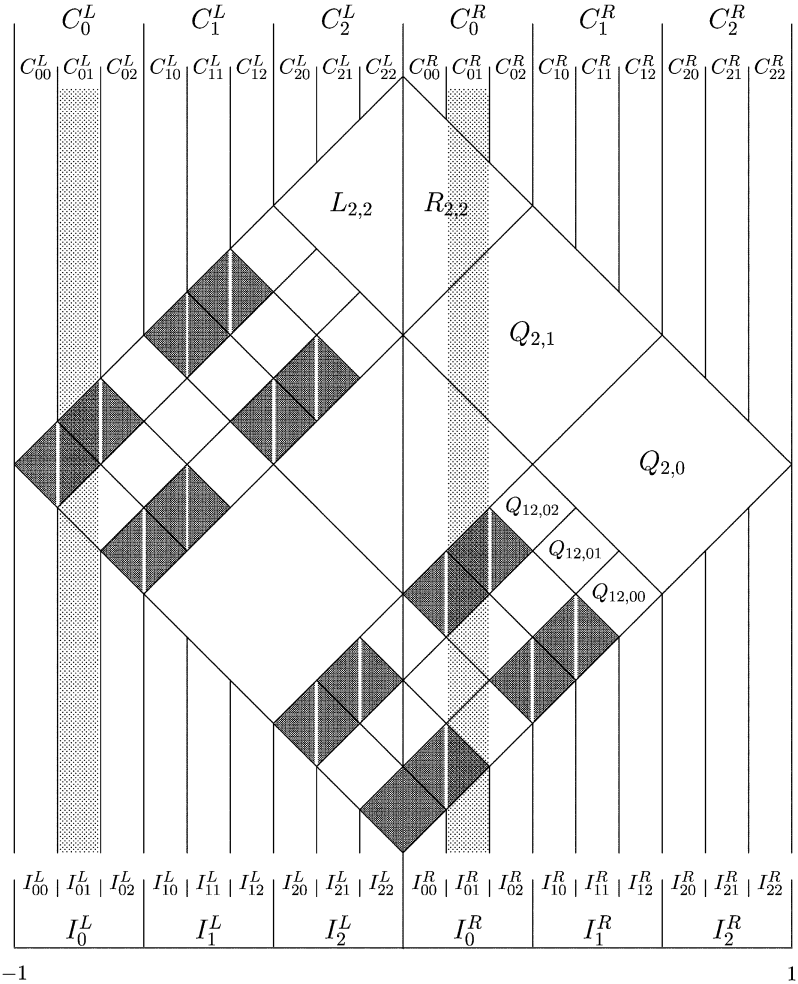

Fig. 2 An illustration for $M = 3$ of the unit square $[0, 1]^2$, scaled and rotated by 45°. The shaded squares form a realization of Λ^2 for 2 out of 3 fractal percolation. The vertical projection gives the ϕ-image $[-1, 5/9]$ of Λ^2

The triangle counts $Z^{UV}(\underline{k}_n)$, with $k_1, k_2, \dots,$ a fixed path, constitute a two type branching process in a varying environment with interaction: the interaction comes from the dependency between triangles that are *aligned*, i.e., triangles contained in respective squares $Q_{i_1\dots i_n, j_1\dots j_n}$ and $Q_{i'_1\dots i'_n, j'_1\dots j'_n}$ with $i_1 \dots i_n = i'_1 \dots i'_n$ or $j_1 \dots j_n = j'_1 \dots j'_n$. Squares that are not aligned will be called *unaligned*.

The *expectation matrices* of the two type branching process are for $\underline{k}_n \in \mathcal{T}$ given by:

$$\mathcal{M}\left(\underline{k}_n\right) := \begin{bmatrix} \mathbb{E}Z^{LL}(\underline{k}_n) & \mathbb{E}Z^{LR}(\underline{k}_n) \\ \mathbb{E}Z^{RL}(\underline{k}_n) & \mathbb{E}Z^{RR}(\underline{k}_n) \end{bmatrix}. \tag{3}$$

These matrices satisfy the basic relation

$$\mathcal{M}(k_1 \ldots k_n) = \mathcal{M}(k_1) \cdots \mathcal{M}(k_n), \tag{4}$$

for all $k_1 \ldots k_n \in \mathcal{T}$.

Lemma 1 shows the importance of the correlation coefficients.

Lemma 1 [1] *For all $k \in \mathbb{A}$ we have*

$$[1 \quad 1]\mathcal{M}(k) = [\mathbb{E}Z^L(k) \quad \mathbb{E}Z^R(k)] = [\gamma_{k+1}\gamma_k]. \tag{5}$$

Proof As in [1] this follows from some careful bookkeeping and

$$\mathbb{P}(Q_{i,j} \subseteq \Lambda^1) = \mathbb{P}\big(I_i \subseteq F_1^1, I_j \subseteq F_2^1\big) = \mathbb{P}_\mu\left(I_i \subseteq F_1^1\right)\mathbb{P}_\lambda\left(I_j \subseteq F_2^1\right) = p_i q_j.$$

□

2.5 Rough Sketch of the Proof of Theorem 1

The idea of the proof is to pair unaligned left and right triangles that survive in the *same* column into what are called Δ-*pairs*.

Suppose we have a Δ-pair in one of the columns with positive probability. If we can prove that there is a strictly positive probability that the number of L-triangles and R-triangles in *all subcolumns* of this column grows exponentially, then it can be shown that with positive probability the M-adic interval corresponding to this column is in the projection $\phi(\Lambda)$. The determining quantity for exponential growth is the smallest correlation coefficient

$$\gamma := \min_{k \in \mathbb{A}} \gamma_k. \tag{6}$$

Now we make use of the fact that conditioned on $\Lambda \neq \emptyset$ the Hausdorff dimension of Λ is almost surely larger than 1, which is implied by $\gamma > 1$.

It can be shown (see [1]) that from this it follows that the number of unaligned squares grows to infinity. By self-similarity of the process each of the unaligned squares has positive probability to generate an interval in the projection, and hence with probability one there will be an interval in the projection.

To show that a Δ-pair occurs somewhere with positive probability it suffices that $\gamma > 1$. So the joint survival condition is only needed to ensure positive probability of exponential growth in all subcolumns of a Δ-pair. For any level l Δ-pair (L^l, R^l) that is contained in a level l column C, the distribution of the number of level $l+n$ V-triangles surviving in Λ^{l+n} in the $\underline{k}_n$-th subcolumn of (L^l, R^l), conditional on the survival of (L^l, R^l) in Λ^l, is independent of l, the particular choice of the column C and the Δ-pair in this column. Therefore, we can unambiguously denote a random variable having this distribution by

$$\tilde{Z}^V(\underline{k}_n) \tag{7}$$

for all $V \in \{L, R\}$ and $\underline{k}_n \in \mathcal{T}$. In general $\tilde{Z}^V(\underline{k}_n)$ does not have the distribution of $Z^V(\underline{k}_n)$ because there is possible dependence between the offspring generation of two level 0 triangles, whereas there is no dependence between the offspring generation of the L-triangle and the R-triangle of a Δ-pair, because they are unaligned by definition of a Δ-pair. However, both do have the same expected value.

In [2] the following lemma on exponential growth of triangles is proved:

Lemma 2 *If $\gamma > 1$, and the joint survival distributions satisfy the joint survival condition, then for all $n \geq 0$*

$$\mathbb{P}(\tilde{Z}^L(\underline{k}_l) \geq \gamma^l, \tilde{Z}^R(\underline{k}_l) \geq \gamma^l \text{ for all } \underline{k}_l \in \mathcal{T}_l \text{ for all } 0 \leq l \leq n) > 0.$$

In Lemma 4 in Sect. 4 we obtain this lemma (with a different growth factor) under weaker conditions than the joint survival condition.

2.6 Higher Order Cantor Sets

The idea of higher order Cantor sets is to collapse n construction steps into one step. Since $\Lambda^n \downarrow \Lambda$ we can for all $n \geq 1$ write

$$\Lambda = \bigcap_{m=1}^{\infty} \Lambda^m = \bigcap_{m=1}^{\infty} \Lambda^{nm}.$$

The sets $(\Lambda^{nm})_{m=1}^{\infty}$ are constructed by joint survival distributions which will be denoted by $\mu^{(n)}$ and $\lambda^{(n)}$. If Theorem 2 fails to answer the interval or not question for the pair (μ, λ), one can hope to get an answer by considering Λ as generated by $(\mu^{(n)}, \lambda^{(n)})$.

The success of this idea is illustrated by Theorem 6.1 in [2], and by Theorem 4. We will also use it for the proof of Lemma 7.

All entities of the nth order random Cantor set will be denoted with a superscript (n). The alphabet now is $\mathbb{A}^{(n)} = \{0, \ldots, M^n - 1\}$ and $\mu^{(n)}$ and $\lambda^{(n)}$ are probability measures on the subsets of $\mathbb{A}^{(n)}$ which are completely determined by μ and λ.

Let us illustrate this with a simple example. Let $M = 2$ and define μ by $\mu(\{0, 1\}) = \mu(\{1\}) = 1/2$. For the corresponding second order Cantor set we have $\mathbb{A}^{(2)} = \{0, 1, 2, 3\}$ and

$$\mu^{(2)}(\{0, 1, 2, 3\}) = \mu^{(2)}(\{1, 2, 3\}) = \mu^{(2)}(\{0, 1, 3\}) = \mu^{(2)}(\{1, 3\}) = \frac{1}{8},$$

$$\mu^{(2)}(\{2, 3\}) = \mu^{(2)}(\{3\}) = \frac{1}{4}.$$

3 The Critical Case

What happens in the critical case when $\gamma = 1$? This was left open in [1] and [2]. Here we will give a simple argument, independent of the other results in this paper, that under some conditions shows that there is almost surely no interval in the difference set. In particular this result permits us to give a complete classification in Theorem 3. We also can tell what happens for critical classical fractal percolation: if $p = 1/\sqrt{M}$, then there is almost surely no interval in the difference set.

Definition 2 The joint survival distributions μ and λ are called *entangled* if for sets $X, Y \subseteq \mathbb{A}$ the inequality $\mu(X)\lambda(Y) > 0$ implies that $X \cap Y \neq \emptyset$.

Proposition 1 *Consider two independent random Cantor sets F_1 and F_2 with joint survival distributions μ and λ having marginal probabilities (p_i) and (q_j), such that $\gamma_0 \leq 1$. Then $F_1 - F_2$ contains no interval a.s., provided that μ and λ are not entangled.*

Proof Let Z_n be the number of 'central' squares in Λ^n, i.e.,

$$Z_n = \#\{i_1 \dots i_n \in \mathcal{T} : Q_{i_1\dots i_n, i_1\dots i_n} \in \Lambda^n\}.$$

Then $Z_0 = 1$, and since these central squares are unaligned, (Z_n) is an ordinary branching process with mean offspring

$$\mathbb{E}[Z_1] = p_0q_0 + p_1q_1 + \dots + p_{M-1}q_{M-1} = \gamma_0 \le 1.$$

Now if $\gamma_0 = 1$, then the offspring distribution is deterministic ($Z_1 \equiv 1$) if and only if μ and λ are entangled ($\mathbb{P}(Z_1 > 0) \ge \mu(X)\lambda(Y) > 0$ if X and Y are sets with $X \cap Y = \emptyset$ and $\mu(X)\lambda(Y) > 0$). Hence, (Z_n) will die out a.s., say at time N. In the sequel we will write the string $i_1 \dots i_n = (k, k, \dots, k)$ for $k \in \mathbb{A}$ as k^n.

Then, because there are no central squares left, $C^R_{0^{N+n}}$ only contains left triangles for all $n \ge 0$. Moreover, the number of left triangles in $(C^R_{0^{N+n}})$ is an ordinary branching process (Y^R_n) with random initial distribution Y^R_0, and mean offspring

$$\mathbb{E}\big[Y^R_1\big] = p_0q_{M-1} \le 1.$$

Similarly, $C^L_{(M-1)^{N+n}}$ only contains right triangles for all $n \ge 0$. Moreover, the number of right triangles in $(C^L_{(M-1)^{N+n}})$ is a branching process (Y^L_n) with Y^L_0, and mean offspring

$$\mathbb{E}\big[Y^L_1\big] = p_{M-1}q_0 \le 1.$$

If both $\mathbb{E}[Y^R_1]$ and $\mathbb{E}[Y^L_1]$ would equal 1, then $p_0q_{M-1} = p_{M-1}q_0 = 1$ and consequently $p_0q_0 = p_{M-1}q_{M-1} = 1$ implying that $\gamma_0 \ge 2$. Hence either $\mathbb{E}[Y^R_1] < 1$ or $\mathbb{E}[Y^L_1] < 1$, such that at least one of the two branching processes (Y^R_n) and (Y^L_n) will die out almost surely, implying that $F_1 - F_2$ has a 'gap' directly left or right of 0. It then follows from selfsimilarity and the denseness of the points $k_1M^{-1} + \dots + k_nM^{-n}$ that $F_1 - F_2$ contains no interval a.s. (cf. [1]). □

That we need at least some restriction on the joint survival distributions in addition to the requirement $\gamma_0 \le 1$ is shown in the following example: Let $M = 2$ and define the joint survival distributions μ and λ by setting $\mu(\{0\}) = \mu(\{1\}) = 1/2$ and $\lambda(\{0, 1\}) = 1$. Then $\gamma = \gamma_0 = 1$, and there is an interval of length 1 in the difference set at a random position. Proposition 1 does not apply since μ and λ are entangled.

4 The Distributed Growth Condition

In this section we introduce a condition for exponential growth of triangles which is based on the following idea: if we can find a column C where we have a sufficient number of Δ-pairs, then under some conditions each of these Δ-pairs can be used to guarantee exponential growth of triangles in a proper subset of the set of subcolumns of C. In some sense we 'spread the burden of proof', and this gives the condition a flexible nature. This is illustrated by the fact that with help of this condition, we can completely classify correlated fractal percolation.

For $X, Y \subseteq \mathbb{A}$ and $e \in \mathbb{A}$ we define $\gamma_e(X, Y)$ to be the eth correlation coefficient corresponding to the joint survival distributions $\mu^\star$ and $\lambda^\star$ assigning probability one to X and Y respectively, i.e.,

$$\gamma_e(X, Y) = \sum_{i \in \mathbb{A}} \mathbf{1}_Y(i) \mathbf{1}_X(i + e). \tag{8}$$

Condition 2 *The pair of joint survival distributions* (μ, λ) *satisfies the* distributed growth condition *if for all* $k \in \mathbb{A}$ *we can find sets* $X_k, Y_k \subseteq \mathbb{A}$ *such that*

(DG0) $\mu(X_k) > 0$ *and* $\lambda(Y_k) > 0$,
(DG1) $\min_{e \in \mathbb{A}} \gamma_e(X_k, Y_k) \geq 1$,
(DG2) $\gamma_k(X_k, Y_k) \geq 2,\ \gamma_{k+1}(X_k, Y_k) \geq 2$.

Lemma 3 *Let* E *denote the event that there exists* $l \geq 1$, $\underline{k}_l \in \mathcal{T}_l$ *and* $U \in \{L, R\}$ *such that* $C^U_{\underline{k}_l}$ *contains at least* M *left and* M *right triangles which are all pairwise unaligned. If the pair of joint survival distributions* (μ, λ) *satisfies the DGC, then*

$$\mathbb{P}(E) > 0.$$

Proof Choose $X_0, Y_0 \subseteq \mathbb{A}$ according to the DGC. Define the joint survival distributions $\mu^\star$ and $\lambda^\star$ by $\mu^\star(X_0) = \lambda^\star(Y_0) = 1$. Then by (DG2) both column sums of the expectation matrix $\mathcal{M}^\star(0)$ are at least 2, implying that

$$[1 \quad 1]\mathcal{M}^\star(0^n) \geq [2^n \quad 2^n],$$

elementwise. The first row of $\mathcal{M}^\star(0^n)$ corresponds to $C^L_{0^n}$, which can contain at most one left triangle and no right triangles. Therefore, both numbers in the second row of $\mathcal{M}^\star(0^n)$ are bounded below by $2^n - 1$. It follows that the numbers of left and right triangles in $C^R_{0^n}$ grow arbitrary large if n is sufficiently large. Since μ and λ assign positive probability to X_0 and Y_0 respectively, the statement of the lemma follows. □

We can now formulate our exponential growth lemma.

Lemma 4 *If the pair of joint survival distributions* (μ, λ) *satisfies the distributed growth condition, then there exist* $l \geq 1$, $\underline{k}_l \in \mathcal{T}_l$ *and* $\eta > 1$ *such that for all* $n \geq 0$

$$\mathbb{P}(Z^L(\underline{k}_l \underline{k}_p) \geq \eta^p, Z^R(\underline{k}_l \underline{k}_p) \geq \eta^p \text{ for all } \underline{k}_p \in \mathcal{T}_p \text{ for all } 0 \leq p \leq n) > 0.$$

Proof Choose $n \geq 0$ arbitrary. For all $k \in \mathbb{A}$ choose $X_k \subseteq \mathbb{A}$ and $Y_k \subseteq \mathbb{A}$ such that these sets satisfy the DGC. Define the joint survival distributions $\mu^\star_k$ and $\lambda^\star_k$ by requiring that $\mu^\star_k(X_k) = \lambda^\star_k(Y_k) = 1$.

Let $k \in \mathbb{A}$ be fixed and consider the expectation matrices corresponding to the triangle growth process defined by $(\mu^\star_k, \lambda^\star_k)$. By (5), their column sums are given by the correlation coefficients corresponding to the pair of joint survival distributions $(\mu^\star_k, \lambda^\star_k)$. So, for all $e \in \mathbb{A}$, both column sums of $\mathcal{M}^\star_k(e)$ are at least 1 and both column sums of $\mathcal{M}^\star_k(k)$ are at least 2. Let p be an integer with $0 \leq p \leq n$. Since for $\underline{k}_p = k_1 \ldots k_p \in \mathcal{T}_p$ we have

$$\mathcal{M}^\star_k(\underline{k}_p) = \mathcal{M}^\star_k(k_1) \ldots \mathcal{M}^\star_k(k_p),$$

it follows that a lower bound for the column sums of $\mathcal{M}_k^\star(\underline{k}_p)$ is determined by the number of k's in the string $\underline{k}_p$. We obtain (omitting the dependence on k, and writing k_j for the jth element in the string $\underline{k}_p$)

$$\gamma^\star_{\underline{k}_p} \geq 2^{\#\{0\leq j\leq p:k_j=k\}}, \qquad \gamma^\star_{\underline{k}_p+1} \geq 2^{\#\{0\leq j\leq p:k_j=k\}}.$$

From the deterministic nature of $\mu_k^\star$ and $\lambda_k^\star$, it follows that the expectation of the number of triangles in some column is simply the number that will occur. This means that for all $0 \leq p \leq n$

$$Z_k^{L;\star}(\underline{k}_p) = \mathbb{E}\big[Z_k^{L;\star}(\underline{k}_p)\big] = \gamma^\star_{\underline{k}_p+1} \geq 2^{\#\{0\leq j\leq p:k_j=k\}},$$
$$Z_k^{R;\star}(\underline{k}_p) = \mathbb{E}\big[Z_k^{R;\star}(\underline{k}_p)\big] = \gamma^\star_{\underline{k}_p} \geq 2^{\#\{0\leq j\leq p:k_j=k\}}.$$

Since (μ, λ) satisfies the DGC, we can by Lemma 3 find an l-adic column $C^U_{\underline{k}_l}$ containing with strictly positive probability at least M left- and M right triangles being all pairwise unaligned. Let this event be denoted by E and abbreviate the notation of this column by C and its subcolumns $C^U_{\underline{k}_l\underline{k}_p}$ by $C_{\underline{k}_p}$.

Now suppose we have a Δ-pair (L, R) in C, in which the growth process behaves according to the pair of joint survival distributions $(\mu_k^\star, \lambda_k^\star)$. Then, for all p and all subcolumns $C_{\underline{k}_p}$ of C, both the number of left and the number of right triangles in $C_{\underline{k}_p} \cap (L \cup R)$ is at least $2^{\#\{0\leq j\leq p:k_j=k\}}$.

Conditional on the event E, we have M left and right triangles in C. We can label them by the elements of $\mathbb{A}$ such that we have M Δ-pairs. These $2M$ triangles are all pairwise unaligned (also if they belong to different Δ-pairs) and hence there is completely no dependence between these triangles. It follows that it is possible that in each of the Δ-pairs the growth process takes place as prescribed by $\mu_k^\star$ and $\lambda_k^\star$, where k is the label of the Δ-pair. Denoting the event that this happens in the first n construction steps after occurrence of E by E_n, we can find a strictly positive lower bound for $\mathbb{P}(E_n|E)$:

$$\mathbb{P}(E_n|E) \geq \prod_{k\in\mathbb{A}} \mu(X_k)^{\sum_{j=1}^n (\#X_k)^{j-1}} \lambda(Y_k)^{\sum_{j=1}^n (\#Y_k)^{j-1}} > 0.$$

Let $0 \leq p \leq n$ and let $C_{\underline{k}_p}$ be an arbitrary M^p-adic subcolumn of C. There must exist a $k = k(\underline{k}_p) \in \mathbb{A}$ such that $\#\{0 \leq j \leq p : k_j = k\} \geq \lceil \frac{p}{M} \rceil$. Hence, given the event E_n, for the numbers of left and right triangles in $C_{\underline{k}_p}$ we have

$$Z^L(\underline{k}_l\underline{k}_p) \geq 2^{\lceil \frac{p}{M} \rceil}, \qquad Z^R(\underline{k}_l\underline{k}_p) \geq 2^{\lceil \frac{p}{M} \rceil}.$$

Taking $\eta = \sqrt[M]{2}$, we obtain

$$\begin{aligned}&\mathbb{P}(Z^L(\underline{k}_l\underline{k}_p) \geq \eta^p, Z^R(\underline{k}_l\underline{k}_p) \geq \eta^p \text{ for all } \underline{k}_p \in \mathcal{T}_p \text{ for all } 0 \leq p \leq n)\\ &\quad\geq \mathbb{P}(E)\mathbb{P}(E_n|E) > 0. \qquad\square\end{aligned}$$

Collecting the results established so far, we can replace the joint survival condition (Condition 1) and Lemma 2 by the distributed growth condition and Lemma 4 to obtain the following useful variation on Theorem 1:

Theorem 2 *Consider two independent random Cantor sets F_1 and F_2 whose joint survival distributions satisfy Condition 2, the* DGC.

(1) *If $\gamma_k > 1$ for all $k \in \mathbb{A}$, then $F_1 - F_2$ contains an interval a.s. on $\{F_1 - F_2 \neq \emptyset\}$.*
(2) *If $\gamma_k < 1, \gamma_{k+1} < 1$ for some $k \in \mathbb{A}$, then $F_1 - F_2$ contains no interval a.s.*

This result is useful since it can be successfully applied to the class of correlated fractal percolation, whilst the JSC is never satisfied for the members of this class. Actually our new condition can always supersede the JSC.

Lemma 5 *Suppose that the joint survival distributions μ and λ satisfy the* JSC. *If $\gamma_k > 1$ for all $k \in \mathbb{A}$, then the pair (μ, λ) satisfies the* DGC.

Proof We take for the sets X_k and Y_k in (8) the marginal supports of μ and λ. Then the JSC implies that (DG0) holds. Since $q_i = 0$ if $i \notin \mathrm{Supp}_m(\lambda)$, and similarly for p_i, we have for all $e \in \mathbb{A}$

$$\begin{aligned}\gamma_e(\mathrm{Supp}_m(\mu), \mathrm{Supp}_m(\lambda)) &= \sum_{i\in\mathbb{A}} \mathbf{1}_{\mathrm{Supp}_m(\lambda)}(i)\mathbf{1}_{\mathrm{Supp}_m(\mu)}(i+e) \\ &\geq \sum_{i\in\mathbb{A}} q_i p_{i+e} = \gamma_e \geq 2,\end{aligned}$$

since the number on the left hand side is an integer larger than 1. Therefore X_k and Y_k certainly satisfy (DG1) and (DG2) for all $k \in \mathbb{A}$. Thus (μ, λ) satisfies the DGC. □

5 Classifying Correlated Fractal Percolation

With the distributed growth condition at our disposal we can make an attempt to solve the Palis problem for correlated fractal percolation. To facilitate our search for sets satisfying the DGC, we introduce an alternative notation for subsets of the alphabet. A subset S of the alphabet $\mathbb{A}$ can be represented as a string of length M with at the ith position a zero or a one, indicating whether or not i is contained in S. For (m, M, p)-percolation, all subsets of $\mathbb{A}$ to which is assigned positive probability correspond to a string consisting of m ones and $M - m$ zeros, where any order of the symbols is allowed. Next we need the notion of the *cyclic shift operator* σ. For any string $X = x_0x_1 \ldots x_{M-2}x_{M-1}$ we define

$$\sigma(X) = x_1x_2 \cdots x_{M-1}x_0. \tag{9}$$

For the kth iterate of σ we use the notation σ^k and for its inverse σ^{-k}. Computing $\gamma_k(X, Y)$ can be done by writing down the two binary strings corresponding to $\sigma^k(X)$ and Y, and then counting in how many positions both strings have a one (this will be called a *coincidence*). This procedure is illustrated in (10) for $M = 9$, $k = 4$ and the sets $X = \{3, 5, 7, 8\}$ and $Y = \{0, 1, 6, 7\}$, where we abuse notation by *also* writing X for the indicator string of X, and similarly for Y (this will never cause confusion).

$$\begin{array}{rccccccccc} X: & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ \sigma^4(X): & 0 & \mathbf{1} & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ Y: & 1 & \mathbf{1} & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array} \tag{10}$$

As we see, there is one coincidence, so $\gamma_4(X, Y) = 1$. Checking the DGC boils down to finding binary strings with the right properties as given in (DG0), (DG1) and (DG2).

Let X and Y be two subsets of the M-adic alphabet $\mathbb{A}$ containing m elements in order to satisfy (DG0). Our strategy is to choose X such that we get a binary string with all ones at the beginning and Y such that the ones are distributed evenly over the string in such a way that at most $m-1$ consecutive zeros occur. This pattern will lead to fulfillment of requirement (DG1). If we have sufficient freedom to choose Y within this framework, then we will also succeed in letting (DG2) be satisfied. The details of this strategy are filled in the proof of the lemma below.

Lemma 6 *For (m, M, p)-percolation the following two assertions hold*:

(1) *If $m < \sqrt{M}$ or $p < \frac{1}{\sqrt{M}}$, then $F_1 - F_2$ contains no interval a.s.*[1]

(2) *If $m \geq \sqrt{M+2}$ and $p > \frac{1}{\sqrt{M}}$, then $F_1 - F_2$ contains an interval a.s. on $\{F_1 - F_2 \neq \emptyset\}$.*

Proof Suppose that $p < \frac{1}{\sqrt{M}}$, then for all $k \in \mathbb{A}$ we have

$$\gamma_k = Mp^2 < M\left(\frac{1}{\sqrt{M}}\right)^2 = 1,$$

and consequently $F_1 - F_2$ contains no interval a.s. by Theorem 2. If $m < \sqrt{M}$, then $p = (1 - \mu(\emptyset))\frac{m}{M} < \frac{1}{\sqrt{M}}$ and consequently the same argument is applicable, completing the proof of the first part of Lemma 6.

For the proof of the second assertion, assume that $m \geq \sqrt{M+2}$ and define $X, Y' \subseteq \mathbb{A}$ by their strings

$$X = 1^m\, 0^{M-m},$$
$$Y' = R[1\, 0^{m-1}]^q,$$

where $q = \lfloor M/m \rfloor$, R is a left substring of $1\, 0^{m-2}$ (R is empty when m divides M), and $[1\, 0^{m-1}]^q$ denotes the string $1\, 0^{m-1}$, q times repeated. Ignoring the trivial case $M = m = 2$ we obtain from $m \geq \sqrt{M+2}$ that we may assume $m \geq 3$.

Since Y' does not contain m consecutive zeros (also cyclically), whereas X begins with m consecutive 1's, we must have

$$\gamma_e(X, Y') \geq 1 \quad \text{for } e = 0, 1, \ldots, M-1.$$

So X and Y' satisfy (DG1). The set X contains m elements, which means that $\mu(X) > 0$.

Note that $q = \lfloor M/m \rfloor$ can not exceed $m-1$, since that would imply $m \leq \sqrt{M}$.

Case 1: $q \leq m-2$ or R is empty.

Then Y' contains at most $m-1$ ones. In order to obtain (DG2), we construct Y'' from Y' by putting a one in the second position (if there is a zero)—note that X and Y'' will then certainly still satisfy (DG1). Moreover, we now have

$$\gamma_0(X, Y'') \geq 2, \qquad \gamma_1(X, Y'') \geq 2,$$

[1] Actually, $m < \sqrt{M}$ implies that $p < 1/\sqrt{M}$. Hence the statement "*If $p < 1/\sqrt{M}$, then $F_1 - F_2$ contains no interval a.s.*" is equivalent to the first assertion of Lemma 6. We formulated the lemma in this way to emphasize what the bounds on m are.

since $m \geq 3$. Finally Y is obtained by adding 1's to Y'' (if necessary) till Y contains m ones—and thus $\mu(Y) > 0$. As an illustration for $M = 7$ and $m = 4$, X is given by 1111000 and (writing $\gamma_k(\cdot)$ for $\gamma_k(X, \cdot)$):

$\cdot$	String	$\mu(\cdot) > 0$	$\gamma_e(\cdot) \geq 1\ \forall e \in \mathbb{A}$	$\gamma_0(\cdot),\gamma_1(\cdot) \geq 2$
Y'	1001000	No	Yes	No
Y''	1101000	No	Yes	Yes
Y	1111000	Yes	Yes	Yes

Now we have found $X_0 := X$ and $Y_0 := Y$ satisfying (DG0), (DG1) and (DG2) for $k = 0$. By observing that

$$\gamma_k(X, \sigma^k Y) = \gamma_0(X, Y); \qquad \gamma_{k+1}(X, \sigma^k Y) = \gamma_1(X, Y), \tag{11}$$

it follows that the DGC holds for any $k \in \mathbb{A}$ if we take $X_k = X$ en $Y_k = \sigma^k Y$.

Case 2: $q = m - 1$ and $R \neq \emptyset$.
Since $m \geq \sqrt{M + 2}$, we have (with r the length of R)

$$M = m^2 - m + r \geq M + 2 - m + r,$$

so $r \leq m - 2$, implying that R does not contain more than $m - 3$ zero's. This gives that $\gamma_0(X, Y) \geq 2$ and $\gamma_1(X, Y) \geq 2$. Now again we can take $X_k = X$ en $Y_k = \sigma^k Y$. Summarizing, for all cases of correlated fractal percolation in part (2) we have shown that (DG0), (DG1) and (DG2) hold. We conclude that the DGC is satisfied.

Moreover, for all $k \in \mathbb{A}$ we find

$$\gamma_k = \sum_{j=0}^{M-1} p_j p_{j+k} = Mp^2 > M\left(\frac{1}{\sqrt{M}}\right)^2 = 1,$$

and therefore, by Theorem 2, $F_1 - F_2$ contains an interval a.s. on $\{F_1 - F_2 \neq \emptyset\}$. □

Lemma 6 still gives no conclusive answer for some combinations of m and M when $p > 1/\sqrt{M}$, namely, those where $m = \sqrt{M + 1}$. By having a look at the 2nd order sets for (m, M, p)-percolation this can be resolved.

Lemma 7 *Consider (m, M, p)-percolation. If $p > \frac{1}{\sqrt{M}}$ and*

$$m = \sqrt{M + 1}, \tag{12}$$

then $F_1 - F_2$ contains an interval a.s. on $\{F_1 - F_2 \neq \emptyset\}$.

Proof First we have a look at the shape of the binary strings corresponding to 2nd order sets to which is assigned positive probability by the 2nd order joint survival distribution $\mu^{(2)}$ of correlated fractal (m, M, p)-percolation. Such a string has length M^2. It should be regarded as consisting of M blocks of length M. Each of these blocks contains either exclusively zeros, or it contains $M - m$ zeros and m ones. Blocks of the latter kind occur exactly m times. Positions in the binary string can be identified with numbers in $\mathbb{A}^{(2)}$: an M^2-adic number represented by $\underline{k}_2 = k_1k_2$ corresponds to the (k_2+1)th position in the (k_1+1)th block.

Note that (12) implies that $M - m(m-1) = m-1$ and $\lfloor M/m \rfloor = m-1$. This means that the two strings X and Y' defined in the proof of Lemma 6 are now equal to (we omit from now on the prime on Y)

$$X = 1^m\, 0^{M-m},$$
$$Y = [1\, 0^{m-2}]\, [1\, 0^{m-1}]^{m-1}.$$

The basic idea of the proof is to replace the 0's in these two strings by blocks 0^M, and the 1's by blocks similar to X or Y to obtain for all $\underline{k}_2 \in \mathbb{A}^{(2)}$ the order 2 strings $X^{(2)}_{\underline{k}_2}$ and $Y^{(2)}_{\underline{k}_2}$ which will satisfy (DG1) and (DG2)—note that by construction (DG0) is then obviously satisfied.

Actually we will replace all the m 1's in X by the string Y. Replacing additionally the $M-m$ 0's by blocks 0^M we obtain $X^{(2)}_{\underline{k}_2}$ independent of $\underline{k}_2$, and hence we will denote it by $X^{(2)}$.

The definition of $Y^{(2)}_{\underline{k}_2}$ is slightly more involved. We first restrict ourselves to the case $k_1 = 0$ and define:

$$Y^{(2)}_{0k_2} := \sigma^{Ms}\big([\sigma^{k_2}(X)\, 0^{(m-2)M}]\, [\sigma^{k_2}(X)\, 0^{(m-1)M}]^{m-1}\big),$$

where s is given by

$$s := \begin{cases} 0 & \text{if } 0 \le k_2 \le m-2, \\ 1 & \text{if } m-1 \le k_2 \le M-1. \end{cases}$$

So the m 1's in Y are replaced by shifted versions of X and 0's by blocks 0^M and finally an additional shift over M positions is applied on the complete string if k_2 is at least $m-1$.

Example 1 *Let $M = 8$ and $m = 3$. Then*

$$X = 11100000 \quad \text{and} \quad Y = 10100100.$$

Writing $O = 0^8$ and $s = \mathbf{1}_{\{n: n\ge 2\}}(k_2)$, we have for $0k_2 \in \mathbb{A}^{(2)}$

$$\begin{array}{lllllllll} X^{(2)} = & Y & Y & Y & O & O & O & O & O \\ Y^{(2)}_{0k_2} = \sigma^{8s}(\sigma^{k_2}(X) & O & \sigma^{k_2}(X) & O & O & \sigma^{k_2}(X) & O & O). \end{array}$$

Suppose that $X^{(2)}$ and $Y^{(2)}_{0k_2}$ satisfy the DGC. Then it is easy to construct sets $X^{(2)}$ and $Y^{(2)}_{k_1k_2}$ satisfying requirements (DG1) and (DG2) for other values of k_1. First observe that all shifted versions of $X^{(2)}$ and $Y^{(2)}_{0k_2}$ still satisfy (DG1). Furthermore we use the fact that

$$\gamma^{(2)}_{\underline{k}_2}\big(X^{(2)}, \sigma^{k_1M}\big(Y^{(2)}_{0k_2}\big)\big) = \gamma^{(2)}_{0k_2}\big(X^{(2)}, Y^{(2)}_{0k_2}\big) \ge 2,$$
$$\gamma^{(2)}_{\underline{k}_2+1}\big(X^{(2)}, \sigma^{k_1M}\big(Y^{(2)}_{0k_2}\big)\big) = \gamma^{(2)}_{(0k_2)+1}\big(X^{(2)}, Y^{(2)}_{0k_2}\big) \ge 2.$$

Now it follows that we can choose $Y^{(2)}_{\underline{k}_2} = \sigma^{k_1M}(Y^{(2)}_{0k_2})$.

To complete the proof, it suffices to check that the sets $X^{(2)}$ and $Y^{(2)}_{0k_2}$ satisfy requirements (DG1) and (DG2) of the DGC. Therefore, we consider the correlation coefficients $\gamma^{(2)}_{\underline{e}_2}(X^{(2)}, Y^{(2)}_{0k_2})$ where $\underline{e}_2 = e_1e_2 \in \mathbb{A}^{(2)}$. We will focus first on the 'coarse' structure, i.e. on those correlation coefficients for which $e_2 = 0$. Here we will always have a string $\sigma^{k_2}(X)$ in $Y^{(2)}_{0k_2}$ coinciding with a string Y in $X^{(2)}$ for the same reason that we always have a coincidence at level 1. This implies that we also always have a string $\sigma^{k_2}(X)$ in $Y^{(2)}_{0k_2}$ coinciding with a zero string of length M in $X^{(2)}$ which is followed (cyclically) by a string Y.

It follows that if we will shift on the 'fine' level by varying e_2, then in all cases we are in the same situation of one $\sigma^{k_2}(X)$ block 'entering' an Y block, and one $\sigma^{k_2}(X)$ 'leaving' an Y block. Thus we get the same coincidences as in the case where $\sigma^{k_2}(X)$ and Y are compared cyclically, and therefore the second order correlation coefficients can be related to the first order correlation coefficients $\gamma_e(\sigma^{k_2}(X), Y)$:

$$\gamma^{(2)}_{\underline{e}_2}\big(X^{(2)}, Y^{(2)}_{0k_2}\big) \geq \gamma_{e_2}(Y, \sigma^{k_2}(X)) \geq 1 \tag{13}$$

for all $\underline{e}_2 = e_1e_2 \in \mathbb{A}^{(2)}$. As we see, (DG1) holds for all $\underline{e}_2 \in \mathbb{A}^{(2)}$.

Now we turn to (DG2). If $e_2 = k_2$, then in (13) we even have by (11) that

$$\gamma^{(2)}_{\underline{e}_2}\big(X^{(2)}, Y^{(2)}_{0k_2}\big) \geq \gamma_{e_2}(Y, \sigma^{k_2}(X)) = \gamma_0(Y, X) = 2,$$

which means that

$$\gamma^{(2)}_{0k_2}\big(X^{(2)}, Y^{(2)}_{0k_2}\big) \geq 2. \tag{14}$$

We still have to check that also $\gamma^{(2)}_{(0k_2)+1}(X^{(2)}, Y^{(2)}_{0k_2}) \geq 2$. First we concentrate on the case where both the first and the last Y-block in $X^{(2)}$ coincide with a $\sigma^{k_2}(X)$ block in $Y^{(2)}_{0k_2}$. To illustrate this in the terms of Example 1, we have:

$$\begin{aligned} X^{(2)} &= \quad Y \qquad Y \qquad Y \qquad O \quad O \qquad O \qquad O \quad O \\ Y^{(2)}_{0k_2} &= \sigma^{k_2}(X) \quad O \quad \sigma^{k_2}(X) \quad O \quad O \quad \sigma^{k_2}(X) \quad O \quad O. \end{aligned}$$

Keeping k_1 fixed to zero and varying k_2, the structure of coincidences we obtain will look like:

YY:						|1	0	**1**	0	0	1	0	0|	1	0	**1**	0	0	1	0	0|			$k_2\downarrow$	$s\downarrow$
$\sigma^{k_2}(X)\sigma^{k_2}(X)$							|1	1	1	0	0	0	0	0|	1	1	1	0	0	0	0	0|		0	0
								|1	1	0	0	0	0	0	1|	1	1	0	0	0	0	0	0|	1	0
	|0	0	0	0	0	0	1	1	|1	0	0	0	0	0	1	1|								2	1
		|0	0	0	0	0	1	1	1	|0	0	0	0	0	1	1	1|						3	1
			|0	0	0	0	1	1	1	0	|0	0	0	0	1	1	1	0|					4	1
				|0	0	0	1	1	1	0	0	|0	0	0	1	1	1	0	0|				5	1
					|0	0	1	1	1	0	0	0	|0	0	1	1	1	0	0	0|			6	1
						|0	1	1	1	0	0	0	0	|0	1	1	1	0	0	0	0|		7	1

Each line in the table corresponds to a value of k_2 and displays the string $\sigma^{k_2}(X)\sigma^{k_2}(X)$. This string is moved over $k_2 + 1$ positions to the right, since we are interested in $\gamma^{(2)}_{(0k_2)+1}(X^{(2)}, Y^{(2)}_{0k_2})$. Then, for each value of k_2 the corresponding value of s (being either 0 or 1) is computed. If $s = 1$, then the string is moved over $M = 8$ positions back to the left. By construction, the number of coincidences of YY with the k_2-line in the table is a lower bound for $\gamma^{(2)}_{(0k_2)+1}(X^{(2)}, Y^{(2)}_{0k_2})$. In each of the lines of the table, we have coincidences with both bold ones in YY. Therefore,

$$\gamma^{(2)}_{(0k_2)+1}\big(X^{(2)}, Y^{(2)}_{0k_2}\big) \geq 2.$$

Combining this with (14), we see that (DG2) holds. Adapting this argument for other values of M and m is straightforward.

As we have seen in the proof of the previous lemma, it is possible to find sufficient independent left and right triangles. Therefore, we have completed our proof that the distributed growth condition is satisfied. We also already saw $p > 1/\sqrt{M}$ implies that $\gamma > 1$, and hence we can use Theorem 2 to finish the proof of Lemma 7. □

Theorem 3 *For correlated fractal* (m, M, p)*-percolation we have*

(1) *If* $\gamma > 1$ *then* $F_1 - F_2$ *contains an interval a.s. on* $\{F_1 - F_2 \neq \emptyset\}$.
(2) *If* $\gamma \leq 1$, *then* $F_1 - F_2$ *contains no interval a.s.*

Proof This result is the combination of Lemma 6, Lemma 7 and Proposition 1. In the latter case we use that $\mu(\emptyset) > 0$ implies that μ is not entangled with itself, and that otherwise $m = \sqrt{M}$ is for $M \geq 4$ smaller or equal to $M/2$, and thus is also not entangled with itself. □

We remark here that since these results will also hold if we merely require that all sets with m elements have positive probability to occur, the theorem will also be true in this more general case.

6 The Lower Spectral Radius in the Symmetric Case

In this section we show that the distributed growth condition propagates to higher order Cantor sets. As a consequence, the spectral radius characterization obtained in [2] can be extended to joint survival distributions satisfying the DGC.

Lemma 8 (*Propagation of the distributed growth condition to higher orders*). *Suppose the pair of joint survival distributions* (μ, λ) *satisfies the* DGC. *Then for all* $n \geq 1$, *the pair of* n*th order joint survival distributions* $(\mu^{(n)}, \lambda^{(n)})$ *satisfies the* DGC.

Proof Choose a string $\underline{k}_n \in \mathbb{A}^{(n)}$, which we write as $\underline{k}_n = kk_2 \ldots k_n$, with $k \in \mathbb{A}$ and $k_2 \ldots k_n \in \mathbb{A}^{(n-1)}$. We check that we can find nth order sets satisfying the DGC for this $\underline{k}_n$. Since the pair (μ, λ) satisfies the DGC, there exist first order sets $X_k, Y_k \subseteq \mathbb{A}$ satisfying (DG0), (DG1) and (DG2). Define

$$X_k^{(n)} := \{\underline{l}_n = l_1 \ldots l_n \in \mathbb{A}^{(n)} : l_j \in X_k \text{ for all } j = 1, \ldots, n\},$$
$$Y_k^{(n)} := \{\underline{l}_n = l_1 \ldots l_n \in \mathbb{A}^{(n)} : l_j \in Y_k \text{ for all } j = 1, \ldots, n\}.$$

Obviously, $\mu^{(n)}(X_k^{(n)}) > 0$ and $\lambda^{(n)}(Y_k^{(n)}) > 0$. Define a new pair of nth order joint survival distributions by $\mu_k^{(n)}(X_k^{(n)}) = \lambda_k^{(n)}(Y_k^{(n)}) = 1$. Also define a first order deterministic pair of joint survival distributions by $\mu_k(X_k) = \lambda_k(Y_k) = 1$. The expectation matrices belonging to these nth order survival distributions are related to those belonging to the first order survival distributions by

$$\mathcal{M}_k^{(n)}(\underline{k}_n) = \mathcal{M}_k(\underline{k}_n) = \mathcal{M}_k(k)\mathcal{M}_k(k_2) \ldots \mathcal{M}_k(k_n).$$

Using that X_k and Y_k satisfy (DG1) and (DG2), and that the columns sums of the expectation matrices are equal to the correlation coefficients, we obtain that

$$[1 \quad 1]\mathcal{M}_k^{(n)}(\underline{k}_n) = [1 \quad 1]\mathcal{M}_k(k)\prod_{j=2}^{n}\mathcal{M}_k(k_j) \geq [2 \quad 2]\prod_{j=2}^{n}\mathcal{M}_k(k_j) \geq [2 \quad 2]$$

elementwise, which means that $Z_k^{(n);L}(\underline{k}_n) \geq 2$ and $Z_k^{(n);R}(\underline{k}_n) \geq 2$, or equivalently

$$\gamma_{\underline{k}_n}\big(X_k^{(n)}, Y_k^{(n)}\big) \geq 2; \qquad \gamma_{\underline{k}_n+1}\big(X_k^{(n)}, Y_k^{(n)}\big) \geq 2.$$

Similarly $\gamma_{\underline{l}_n}(X_k^{(n)}, Y_k^{(n)}) \geq 1$ for all $\underline{l}_n \in \mathbb{A}^{(n)}$. It follows that the pair $(\mu^{(n)}, \lambda^{(n)})$ satisfies the DGC. □

This propagation property leads to the theorem below. The lower spectral radius $\underline{\rho}(\Sigma)$ of a set Σ of square matrices is defined by

$$\underline{\rho}(\Sigma) := \liminf_{n\to\infty} \min_{A_1,\ldots,A_n\in\Sigma} \|A_1 \ldots A_n\|^{1/n},$$

for some matrix norm $\|\cdot\|$. For two M-adic random Cantor sets, let Σ_M be the corresponding collection of expectation matrices

$$\Sigma_M := \{\mathcal{M}(0), \ldots, \mathcal{M}(M-1)\}. \tag{15}$$

Then we obtain the following result:

Theorem 4 *Consider the algebraic difference $F_1 - F_2$ between two M-adic independent random Cantor sets F_1 and F_2 with the same joint survival distribution satisfying the distributed growth condition.*

(1) *If $\underline{\rho}(\Sigma_M) > 1$, then $F_1 - F_2$ contains no interval a.s. on $\{F_1 - F_2 \neq \emptyset\}$.*
(2) *If $\underline{\rho}(\Sigma_M) < 1$, then $F_1 - F_2$ contains no interval a.s.*

Proof The proof is basically the same as the proof of Theorem 6.1 in [2]. There is a difference in the fact that here we do not require irreducibility explicitly. From the symmetry $\mu = \lambda$ it follows that $m_e = m_{-e}$ for all $e \in \mathbb{A} \cup -\mathbb{A}$. Now, since the DGC holds, we get the irreducibility for free.

After derivation of the same statements concerning the nth order correlation coefficients as in [2], we apply our Theorem 2. This is justified by the fact that the DGC propagates to higher orders, as was shown in Lemma 8. □

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